Metric Spaces and Topology
Lecture 20

Continuous functions. For top spaces X and Y, a function $f: X \to Y$ is called *continuous* at a pt. $x \in X$ if the f-preimage of any neighbourhood of $f(x)$ is a (not necessarily open) neighbourhood of x. (Recall that a *neighbourhood* of a pt $x \in X$ is a set U s.t. $x \in \text{int}(U) \Leftrightarrow \exists$ open $U' \subseteq U$ s.t. $x \in U'$.) f is called *continuous* if it's continuous at every point of X ($\Leftrightarrow$ f-preimages of open sets are open).

Prop. Let $\mathcal{B}$ be a prebasis for the top. of Y. A function $f: X \to Y$ is continuous $\Leftrightarrow$ the f-preimages of sets in $\mathcal{B}$ are open.

Proof. For $\Leftarrow$, note that if preimages of sets $V_1, \dots, V_n$ are open, then $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$ is also open. Furthermore, if the preimages of many sets are open, so is the preimage of their union since f^{-1} commutes with unions as well. $\square$

Example. To check that a function $f: X \to \mathbb{R}$ is continuous, it's enough to check that the preimages of $(-\infty, a)$ and $(b, +\infty)$ are open, for all a,b in some dense subset $D \subseteq \mathbb{R}$.

A function $f: X \to Y$ is called a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous, i.e. f maps open to open and vice versa. $f: X \to Y$ is called an **embedding** if it is a homeomorphism from X to its image $f(X)$ in the relative top of Y.

<u>Warning.</u> Being continuous and injective is not enough to be an embedding. For example:

(a) $id: (\mathbb{R}, \text{discrete top}) \to (\mathbb{R}, \text{Euclidean})$.

(b) let $f: 2^{\mathbb{N}} \twoheadrightarrow [0,1]$ the usual surjection.

$$x \mapsto \underbrace{0.x_0 x_1 x_2}_{\text{binary rep.}}$$

This is not injective because $f(w10^\infty) = f(w01^\infty)$. But such points for a ctbl (dense) set D, so the restriction $f|_{2^{\mathbb{N}} \setminus D} : 2^{\mathbb{N}} \setminus D \to [0,1]$ is a continuous bijection. However, f^{-1} is not continuous because f doesn't map clopen to clopen sets (there are many clopen sets in $2^{\mathbb{N}} \setminus D$, while only $[0,1]$ and $\emptyset$ are clopen in $[0,1]$).

Def. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a family of functions $f_i : X \to Y_i$, where X is a set and Y_i is a top. space. The top. on X generated by $\mathcal{F}$ is the coarsest top. on X that makes all functions f_i, $i \in I$, continuous. (Such a top. exists because intersection of topologies is a topology.)

Prop. The top gen. by $\mathcal{F}$ is the same as the top generated by the sets $f_i^{-1}(V_i)$ where $V_i \subseteq Y_i$ is open and $i \in I$.

Product topology (= pointwise convergence topology).

Let $(X_i)_{i \in I}$ be a possibly (unctbl) sequence of top. spaces. Consider the product of this:

$$X := \prod_{i \in I} X_i$$

which consist of sequences $(x_i)_{i \in I}$ s.t. $x_i \in X_i$.

$X \neq \emptyset \overset{AC}{\Leftrightarrow} \forall i \in I,\ X_i \neq \emptyset$. We'd like to equip X with a natural and useful topology. If I is finite, then, $X := X_1 \times X_2 \times \dots \times X_n$ and, like with $\mathbb{R}^n$, the natural top. on X is generated by rectangles $U_1 \times U_2 \times \dots \times U_n$, where each $U_i \subseteq X_i$ is open. When I is infinite, should we take the sets of the form $\prod_{i \in I} U_i$ as open, where $U_i \subseteq X_i$ is open for all $i \in I$?

In the case where I is finite, the top. generated by sets $U_1 \times \dots \times U_n$ is exactly the top where:

(i) a sequence in X converges $\Leftrightarrow$ it converges in every coordinate.

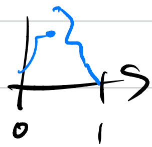
(ii) the projection $proj_i$ onto each coordinate i is continuous. In fact, this top is exactly the top. generated by $\{proj_1, proj_2, \dots, proj_n\}$. Indeed, this top. is generated by sets $proj_i^{-1}(U_i) = X_1 \times \dots \times X_{i-1} \times U_i \times X_{i+1} \times \dots \times X_n$ for $i \in I$ & $U_i \subseteq X_i$ open. Hence the sets $U_1 \times \dots \times U_n$ form a basis being finite intersections of prebasic open sets.

We would like the product top on X even for unctbl I

to satisfy (i) and also (ii) if possible.

Examples. (a) In $A^{\mathbb{N}}$, the usual top. generated by the cylinders $[w]$ is such that a sequence $(x_n) \subseteq A^{\mathbb{N}}$ converges $\Leftrightarrow$ $\forall i \in \mathbb{N}$ the sequence $(x_n(i))$ converges ($\Leftrightarrow$ is eventually constant since the top. on A is discrete).

(b) Let $X := \mathbb{R}^{[0,1]}$ = the set of all functions $[0,1] \to \mathbb{R}$.
$= \prod_{i \in [0,1]} \mathbb{R}_i$, where $\mathbb{R}_i = \mathbb{R}$ for all $i \in [0,1]$.

Each element $f \in X$ is a function 
We would like to equip X with a top that captures pointwise converges, i.e.
$f_n \to f \Leftrightarrow \forall x \in [0,1],\ f_n(x) \to f(x)$.
The top. of uniform metric is too fine (has too many open sets) so only $\Rightarrow$ direction holds but $\Leftarrow$ fails:

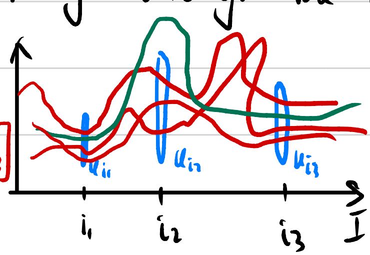
Example. Let $f_n(x) := x^n$, $f_n : [0,1] \to [0,1]$
then $f_n \to f$ pointwise where
$f(x) := \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$, but $f_n \not\to_{\text{uniformly}} f$. $\boxed{\text{HW}}$

Nevertheless, $\forall x \in [0,1]$, $f_n(x) = x^n \to f(x)$.

The idea is to define the product top. using (ii) and then prove that (i) holds.

Def. The product top. on $X := \prod_{i \in I} X_i$ is the coarsest top make each projection $proj_i : X \to X_i$ continuous.
In other words, it is generated by cylinders of the form $[i \mapsto U_i] := \{ x \in X : x_i \in U_i \}$,
for $i \in I$ and $U_i \subseteq X_i$ open.
The basis for this top will then be formed by the cylindrical sets:
$[i_1 \mapsto U_i, i_2 \mapsto U_2, \dots, i_n \mapsto U_n] := \{ x \in X : \forall j \leq n, x(i_j) \in U_j \}$.

For example, if $I := \mathbb{N}$, then the following sets form a basis:
$$U_0 \times U_1 \times \dots \times U_n \times X_{n+1} \times X_{n+2} \times \dots$$
Picture the cylinders $[i_1 \mapsto U_1, \dots, i_n \mapsto U_n]$ as the sets of all snakes (= function graphs) that go through the finitely many loops $U_{i_1}, U_{i_2}, \dots, U_{i_n}$:



let $X_i := Y$ $\forall i \in I$

$[i_1 \mapsto U_{i_1}, i_2 \mapsto U_{i_2}, i_3 \mapsto U_{i_3}]$

Remark. The top. on X generated by the rectangles/boxes:
$$\prod_{i\in I} U_i, \text{ where } U_i \subseteq X_i \text{ is open,}$$
is called the box topology and when I is infinite, it is too fine to be useful. For example, the box top. on $A^{\mathbb{N}}$ is the discrete top since every singleton is a box.

Prop. The product top indeed captures pointwise convergence, i.e. for a sequence $(f_n) \subseteq X := \prod_{i\in I} X_i$ and $f \in X$,
$f_n \to f$ in the product top $\iff$ $\forall i \in I$, $f_n(i) \to f(i)$.

Proof. $\Leftarrow$. Let $V \ni f$ be a basic open neighb. of f, i.e. a cylinder $[i_1 \mapsto U_{i_1}, i_2 \mapsto U_{i_2}, \dots, i_m \mapsto U_{i_m}]$, where $U_{i_j} \subseteq X_{i_j}$ is an open neighbourhood of $f(i_j)$.
We know that $\forall i := i_1, i_2, \dots, i_m$, $f_n(i) \to f(i)$, hence $\forall^\infty n$ $f_n(i) \in U_i$. Thus, by taking the max of n that works for each $i = i_1, \dots, i_m$, we get that
$\forall^\infty n$ $\forall i \in \{i_1, \dots, i_m\}$, $f_n(i) \in U_i$, i.e.
$\forall^\infty n$ $f_n \in [i_1 \mapsto U_{i_1}, \dots, i_m \mapsto U_{i_m}]$.

$\Rightarrow$. Easier. $\boxed{\text{HW}}$ $\square$